Name: ____________________

A-Level Mathematics — Practice Exam

Instructions: Answer all questions in the spaces provided. Show your working — method marks are awarded even when the final answer is wrong. The answer key with worked solutions is available at examtex.com/free-exams.

Time Limit: 75 minutes

Calculations (8 questions)

[4 points] 1. Indices and surds.

[2 points] (a) Simplify $\frac{8^{\frac{2}{3}} \times 4^{-\frac{1}{2}}}{2^{-3}}$, giving your answer as an integer.

[2 points] (b) Rationalise the denominator of $\frac{14}{3-\sqrt{2}}$, giving your answer in the form $a + b\sqrt{2}$, where a and b are integers.

[5 points] 2. The function f is defined by $f(x) = 2x^2 - 12x + 23$.

$f(x) = 2x^2 - 12x + 23$

[2 points] (a) Express $f(x)$ in the form $a(x + b)^2 + c$, where a, b and c are integers to be found.

[1 point] (b) Hence write down the coordinates of the vertex of the curve $y = f(x)$.

[2 points] (c) Using the discriminant, determine the number of real roots of the equation $f(x) = 0$.

[6 points] 3. The circle C has centre $(3, -2)$ and passes through the point $P(7, 1)$.

[3 points] (a) Find an equation for C.

[3 points] (b) Find an equation of the tangent to C at the point P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[5 points] 4. This question is about the binomial expansion of $(2 + 3x)^5$.

$(2 + 3x)^5$

[3 points] (a) Find the first four terms, in ascending powers of x, of the binomial expansion of $(2 + 3x)^5$. Give each coefficient as an integer.

[2 points] (b) Use your expansion with a suitable value of x to estimate 2.03^5, giving your answer to 4 decimal places.

[7 points] 5. A curve has equation $y = 2x^3 - 9x^2 + 12x - 3$.

$y = 2x^3 - 9x^2 + 12x - 3$

[4 points] (a) Find the coordinates of the stationary points of the curve.

[2 points] (b) Determine the nature of each stationary point.

[1 point] (c) State the range of values of x for which y is increasing.

[6 points] 6. The curve with equation $y = 4x - x^2$ and the line with equation $y = x$ enclose a finite region R. (For a sketch: the curve is an inverted parabola through the origin with maximum at $(2, 4)$; the line passes through the origin with gradient 1 and cuts the curve again to the right, with the curve above the line between the two intersections.)

$y = 4x - x^2, \qquad y = x$

[2 points] (a) Find the coordinates of the points of intersection of the curve and the line.

[4 points] (b) Find the exact area of the region R.

[6 points] 7. Solve the equation $2\sin^2\theta + 3\cos\theta = 3$ for $0^\circ \leq \theta \leq 360^\circ$. Show each step of your working.

$$2\sin^2\theta + 3\cos\theta = 3, \qquad 0^\circ \leq \theta \leq 360^\circ$$

[8 points] 8. Statistics. The discrete random variable X has the probability distribution $P(X = 1) = 0.1$, $P(X = 2) = 0.3$, $P(X = 3) = k$, $P(X = 4) = 0.2$, where k is a constant.

$$P(X = 1) = 0.1, \quad P(X = 2) = 0.3, \quad P(X = 3) = k, \quad P(X = 4) = 0.2$$

[1 point] (a) Find the value of k.

[4 points] (b) Find $E(X)$ and $\mathrm{Var}(X)$.

[3 points] (c) Five independent observations of X are taken. Find the probability that exactly two of them take the value 4.

Proofs (1 question)

[4 points] 9. Prove that, for every integer n, $(2n+1)^2 - 1$ is divisible by 8.

Answer Key

1. (a) Write every term as a power of 2: $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2$ and $4^{-\frac{1}{2}} = (2^2)^{-\frac{1}{2}} = 2^{-1}$, so the expression is $\frac{2^2 \times 2^{-1}}{2^{-3}} = 2^{2-1+3} = 2^4 = 16$. (M1: all terms expressed as powers of 2 and index laws applied; A1: 16.)

(b) Multiply numerator and denominator by the conjugate $3 + \sqrt{2}$: $\frac{14(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})} = \frac{14(3+\sqrt{2})}{9-2} = \frac{14(3+\sqrt{2})}{7} = 2(3 + \sqrt{2}) = 6 + 2\sqrt{2}$, so $a = 6$, $b = 2$. (M1: multiply by the conjugate and obtain denominator 7; A1: $6 + 2\sqrt{2}$.)

2. (a) Take out the factor of 2 from the x terms: $f(x) = 2(x^2 - 6x) + 23 = 2\left[(x-3)^2 - 9\right] + 23 = 2(x-3)^2 - 18 + 23 = 2(x-3)^2 + 5$. So $a = 2$, $b = -3$, $c = 5$. (M1: attempt to complete the square with the 2 correctly factored out; A1: $2(x-3)^2 + 5$.)

(b) From the completed-square form, the minimum of $2(x-3)^2 + 5$ occurs when the bracket is zero: the vertex is $(3, 5)$. (B1: $(3, 5)$ — a minimum point, since $a = 2 > 0$.)

(c) Discriminant $= b^2 - 4ac = (-12)^2 - 4(2)(23) = 144 - 184 = -40$. Since $-40 < 0$, the equation $f(x) = 0$ has no real roots — consistent with part (b), because the minimum value of f is $5 > 0$, so the curve never meets the x-axis. (M1: correct substitution into $b^2 - 4ac$; A1: $-40 < 0$, so no real roots.)

3. (a) The radius is the distance from the centre to P: $r^2 = (7-3)^2 + (1-(-2))^2 = 4^2 + 3^2 = 16 + 9 = 25$. So the equation of C is $(x-3)^2 + (y+2)^2 = 25$. (M1: attempt at r^2 using the distance between centre and P; A1: $r^2 = 25$; A1: correct circle equation, including $(y+2)^2$ for the negative y-coordinate.)

(b) Gradient of the radius CP: $\frac{1-(-2)}{7-3} = \frac{3}{4}$. The tangent is perpendicular to the radius, so its gradient is $-\frac{4}{3}$. Through $P(7, 1)$: $y - 1 = -\frac{4}{3}(x - 7)$, so $3y - 3 = -4x + 28$, giving $4x + 3y - 31 = 0$. (M1: gradient of radius; M1: negative reciprocal used for the tangent gradient; A1: $4x + 3y - 31 = 0$.)

4. (a) Using $\binom{5}{k}2^{5-k}(3x)^k$: the terms are $\binom{5}{0}2^5 = 32$; $\binom{5}{1}2^4(3x) = 5 \times 16 \times 3x = 240x$; $\binom{5}{2}2^3(3x)^2 = 10 \times 8 \times 9x^2 = 720x^2$; $\binom{5}{3}2^2(3x)^3 = 10 \times 4 \times 27x^3 = 1080x^3$. So $(2+3x)^5 = 32 + 240x + 720x^2 + 1080x^3 + \ldots$ (M1: correct structure with coefficients 1, 5, 10, 10 and decreasing powers of 2; A1: two terms correct; A1: all four terms correct.)

(b) Since $2 + 3x = 2.03$ when $x = 0.01$, substitute $x = 0.01$: $32 + 240(0.01) + 720(0.01)^2 + 1080(0.01)^3 = 32 + 2.4 + 0.072 + 0.00108 = 34.47308$. So $2.03^5 \approx 34.4731$ (4 d.p.). (M1: identify $x = 0.01$ and substitute into the expansion; A1: 34.4731 — the true value is 34.47309 to 5 d.p., so the truncated expansion is accurate to all 4 decimal places.)

5. (a) $\frac{dy}{dx} = 6x^2 - 18x + 12$. At a stationary point $\frac{dy}{dx} = 0$: $6(x^2 - 3x + 2) = 0$, so $6(x-1)(x-2) = 0$, giving $x = 1$ or $x = 2$. Then $y(1) = 2 - 9 + 12 - 3 = 2$ and $y(2) = 16 - 36 + 24 - 3 = 1$. The stationary points are $(1, 2)$ and $(2, 1)$. (M1: attempt to differentiate; A1: correct $\frac{dy}{dx}$; M1: set derivative to zero and solve the quadratic; A1: both points correct.)

(b) $\frac{d^2y}{dx^2} = 12x - 18$. At $x = 1$: $12 - 18 = -6 < 0$, so $(1, 2)$ is a local maximum. At $x = 2$: $24 - 18 = 6 > 0$, so $(2, 1)$ is a local minimum. (M1: evaluate the second derivative at both x-values; A1: maximum at $(1, 2)$, minimum at $(2, 1)$, with the sign of $\frac{d^2y}{dx^2}$ stated.)

(c) y is increasing where $\frac{dy}{dx} = 6(x-1)(x-2) > 0$, i.e. outside the roots of the positive quadratic: $x < 1$ or $x > 2$. (B1: both intervals.)

6. (a) Set the equations equal: $4x - x^2 = x$, so $3x - x^2 = 0$, i.e. $x(3-x) = 0$, giving $x = 0$ or $x = 3$. The points of intersection are $(0,0)$ and $(3,3)$. (M1: equate and factorise; A1: both points.)

(b) Between $x = 0$ and $x = 3$ the curve lies above the line (e.g. at $x = 1$: curve = 3, line = 1), so the area is $\int_0^3 \left[(4x - x^2) - x\right]\, dx = \int_0^3 (3x - x^2)\, dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - 9 = \frac{9}{2}$. (M1: area set up as the integral of curve minus line between their intersections; A1: correct integrand $3x - x^2$; A1: correct integration; A1: exact area $\frac{9}{2}$.)

7. Use $\sin^2\theta = 1 - \cos^2\theta$: $2(1 - \cos^2\theta) + 3\cos\theta = 3$, so $2 - 2\cos^2\theta + 3\cos\theta = 3$, which rearranges to $2\cos^2\theta - 3\cos\theta + 1 = 0$. Factorise: $(2\cos\theta - 1)(\cos\theta - 1) = 0$, so $\cos\theta = \frac{1}{2}$ or $\cos\theta = 1$. For $\cos\theta = \frac{1}{2}$: $\theta = 60^\circ$ or $\theta = 360^\circ - 60^\circ = 300^\circ$. For $\cos\theta = 1$: $\theta = 0^\circ$ or $\theta = 360^\circ$ (both endpoints are in the closed interval). The solutions are $\theta = 0^\circ, 60^\circ, 300^\circ, 360^\circ$. Check, e.g. $\theta = 300^\circ$: $2\left(\frac{3}{4}\right) + 3\left(\frac{1}{2}\right) = \frac{3}{2} + \frac{3}{2} = 3$, as required. (M1: use of $\sin^2\theta + \cos^2\theta = 1$; A1: correct quadratic in $\cos\theta$; M1: factorise or use the quadratic formula; A1: both values of $\cos\theta$; A1: 60° and 300°; A1: 0° and 360° — losing the endpoint solutions is the most common error.)

8. (a) Probabilities sum to 1: $k = 1 - (0.1 + 0.3 + 0.2) = 0.4$. (B1: $k = 0.4$.)

(b) $E(X) = \sum x\,P(X = x) = 1(0.1) + 2(0.3) + 3(0.4) + 4(0.2) = 0.1 + 0.6 + 1.2 + 0.8 = 2.7$. $E(X^2) = 1(0.1) + 4(0.3) + 9(0.4) + 16(0.2) = 0.1 + 1.2 + 3.6 + 3.2 = 8.1$. $\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 8.1 - 2.7^2 = 8.1 - 7.29 = 0.81$. (M1: correct method for $E(X)$; A1: $E(X) = 2.7$; M1: use of $E(X^2) - [E(X)]^2$; A1: $\mathrm{Var}(X) = 0.81$.)

(c) Let Y be the number of observations equal to 4. Each observation is 4 with probability 0.2, independently, so $Y \sim B(5, 0.2)$. Then $P(Y = 2) = \binom{5}{2}(0.2)^2(0.8)^3 = 10 \times 0.04 \times 0.512 = 0.2048$. (B1: binomial model $B(5, 0.2)$ stated; M1: correct binomial probability structure with $\binom{5}{2}$; A1: 0.2048.)

9. Expand: $(2n+1)^2 - 1 = 4n^2 + 4n + 1 - 1 = 4n^2 + 4n = 4n(n+1)$. Now n and $n+1$ are consecutive integers, so exactly one of them is even; hence $n(n+1) = 2m$ for some integer m. Therefore $(2n+1)^2 - 1 = 4 \times 2m = 8m$, which is divisible by 8 for every integer n, completing the proof. (M1: expand the bracket correctly; A1: factorised form $4n(n+1)$; M1: argue that the product of consecutive integers is even; A1: complete conclusion $8m$, with the argument holding for all integers n, including negatives and zero.)

Worked Solutions

Question 1. *(a)* Write every term as a power of 2: $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2$ and $4^{-\frac{1}{2}} = (2^2)^{-\frac{1}{2}} = 2^{-1}$, so the expression is $\frac{2^2 \times 2^{-1}}{2^{-3}} = 2^{2-1+3} = 2^4 = 16$. (M1: all terms expressed as powers of 2 and index laws applied; A1: 16.)

(b) Multiply numerator and denominator by the conjugate $3 + \sqrt{2}$: $\frac{14(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})} = \frac{14(3+\sqrt{2})}{9-2} = \frac{14(3+\sqrt{2})}{7} = 2(3+\sqrt{2}) = 6 + 2\sqrt{2}$, so $a = 6$, $b = 2$. (M1: multiply by the conjugate and obtain denominator 7; A1: $6 + 2\sqrt{2}$.)

Question 2. *(a)* Take out the factor of 2 from the x terms: $f(x) = 2(x^2-6x)+23 = 2\left[(x-3)^2 - 9\right] + 23 = 2(x-3)^2 - 18 + 23 = 2(x-3)^2 + 5$. So $a = 2$, $b = -3$, $c = 5$. (M1: attempt to complete the square with the 2 correctly factored out; A1: $2(x-3)^2 + 5$.)

(b) From the completed-square form, the minimum of $2(x-3)^2 + 5$ occurs when the bracket is zero: the vertex is $(3, 5)$. (B1: $(3, 5)$ — a minimum point, since $a = 2 > 0$.)

(c) Discriminant $= b^2 - 4ac = (-12)^2 - 4(2)(23) = 144 - 184 = -40$. Since $-40 < 0$, the equation $f(x) = 0$ has no real roots — consistent with part (b), because the minimum value of f is $5 > 0$, so the curve never meets the x-axis. (M1: correct substitution into $b^2 - 4ac$; A1: $-40 < 0$, so no real roots.)

Question 3. *(a)* The radius is the distance from the centre to P: $r^2 = (7-3)^2 + (1-(-2))^2 = 4^2 + 3^2 = 16 + 9 = 25$. So the equation of C is $(x-3)^2 + (y+2)^2 = 25$. (M1: attempt at r^2 using the distance between centre and P; A1: $r^2 = 25$; A1: correct circle equation, including $(y+2)^2$ for the negative y-coordinate.)

(b) Gradient of the radius CP: $\frac{1-(-2)}{7-3} = \frac{3}{4}$. The tangent is perpendicular to the radius, so its gradient is $-\frac{4}{3}$. Through $P(7, 1)$: $y - 1 = -\frac{4}{3}(x-7)$, so $3y - 3 = -4x + 28$, giving $4x + 3y - 31 = 0$. (M1: gradient of radius; M1: negative reciprocal used for the tangent gradient; A1: $4x + 3y - 31 = 0$.)

Question 4. *(a)* Using $\binom{5}{k}2^{5-k}(3x)^k$: the terms are $\binom{5}{0}2^5 = 32$; $\binom{5}{1}2^4(3x) = 5 \times 16 \times 3x = 240x$; $\binom{5}{2}2^3(3x)^2 = 10 \times 8 \times 9x^2 = 720x^2$; $\binom{5}{3}2^2(3x)^3 = 10 \times 4 \times 27x^3 = 1080x^3$. So $(2+3x)^5 = 32 + 240x + 720x^2 + 1080x^3 + \ldots$ (M1: correct structure with coefficients 1, 5, 10, 10 and decreasing powers of 2; A1: two terms correct; A1: all four terms correct.)

(b) Since $2 + 3x = 2.03$ when $x = 0.01$, substitute $x = 0.01$: $32 + 240(0.01) + 720(0.01)^2 + 1080(0.01)^3 = 32 + 2.4 + 0.072 + 0.00108 = 34.47308$. So $2.03^5 \approx 34.4731$ (4 d.p.). (M1: identify $x = 0.01$ and substitute into the expansion; A1: 34.4731 — the true value is 34.47309 to 5 d.p., so the truncated expansion is accurate to all 4 decimal places.)

Question 5. *(a)* $\frac{dy}{dx} = 6x^2 - 18x + 12$. At a stationary point $\frac{dy}{dx} = 0$: $6(x^2 - 3x + 2) = 0$, so $6(x-1)(x-2) = 0$, giving $x = 1$ or $x = 2$. Then $y(1) = 2 - 9 + 12 - 3 = 2$ and $y(2) = 16 - 36 + 24 - 3 = 1$. The stationary points are $(1, 2)$ and $(2, 1)$. (M1: attempt to differentiate; A1: correct $\frac{dy}{dx}$; M1: set derivative to zero and solve the quadratic; A1: both points correct.)

(b) $\frac{d^2y}{dx^2} = 12x - 18$. At $x = 1$: $12 - 18 = -6 < 0$, so $(1, 2)$ is a local maximum. At $x = 2$: $24 - 18 = 6 > 0$, so $(2, 1)$ is a local minimum. (M1: evaluate the second derivative at both x-values; A1: maximum at $(1, 2)$, minimum at $(2, 1)$, with the sign of $\frac{d^2y}{dx^2}$ stated.)

(c) y is increasing where $\frac{dy}{dx} = 6(x-1)(x-2) > 0$, i.e. outside the roots of the positive quadratic: $x < 1$ or $x > 2$. (B1: both intervals.)

Question 6. *(a)* Set the equations equal: $4x - x^2 = x$, so $3x - x^2 = 0$, i.e. $x(3-x) = 0$, giving $x = 0$ or $x = 3$. The points of intersection are $(0,0)$ and $(3,3)$. (M1: equate and factorise; A1: both points.)

(b) Between $x = 0$ and $x = 3$ the curve lies above the line (e.g. at $x = 1$: curve $= 3$, line $= 1$), so the area is $\int_0^3 \left[(4x - x^2) - x\right] \, dx = \int_0^3 (3x - x^2)\, dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - 9 = \frac{9}{2}$. (M1: area set up as the integral of curve minus line between their intersections; A1: correct integrand $3x - x^2$; A1: correct integration; A1: exact area $\frac{9}{2}$.)

Question 7. Use $\sin^2\theta = 1 - \cos^2\theta$: $2(1 - \cos^2\theta) + 3\cos\theta = 3$, so $2 - 2\cos^2\theta + 3\cos\theta = 3$, which rearranges to $2\cos^2\theta - 3\cos\theta + 1 = 0$. Factorise: $(2\cos\theta - 1)(\cos\theta - 1) = 0$, so $\cos\theta = \frac{1}{2}$ or $\cos\theta = 1$. For $\cos\theta = \frac{1}{2}$: $\theta = 60^\circ$ or $\theta = 360^\circ - 60^\circ = 300^\circ$. For $\cos\theta = 1$: $\theta = 0^\circ$ or $\theta = 360^\circ$ (both endpoints are in the closed interval). The solutions are $\theta = 0^\circ, 60^\circ, 300^\circ, 360^\circ$. Check, e.g. $\theta = 300^\circ$: $2\left(\frac{3}{4}\right) + 3\left(\frac{1}{2}\right) = \frac{3}{2} + \frac{3}{2} = 3$, as required. (M1: use of $\sin^2\theta + \cos^2\theta = 1$; A1: correct quadratic in $\cos\theta$; M1: factorise or use the quadratic formula; A1: both values of $\cos\theta$; A1: 60° and 300°; A1: 0° and 360° — losing the endpoint solutions is the most common error.)

Question 8. *(a)* Probabilities sum to 1: $k = 1 - (0.1 + 0.3 + 0.2) = 0.4$. (B1: $k = 0.4$.)

(b) $E(X) = \sum x\, P(X = x) = 1(0.1) + 2(0.3) + 3(0.4) + 4(0.2) = 0.1 + 0.6 + 1.2 + 0.8 = 2.7$. $E(X^2) = 1(0.1) + 4(0.3) + 9(0.4) + 16(0.2) = 0.1 + 1.2 + 3.6 + 3.2 = 8.1$. $\text{Var}(X) = E(X^2) - [E(X)]^2 = 8.1 - 2.7^2 = 8.1 - 7.29 = 0.81$. (M1: correct method for $E(X)$; A1: $E(X) = 2.7$; M1: use of $E(X^2) - [E(X)]^2$; A1: $\text{Var}(X) = 0.81$.)

(c) Let Y be the number of observations equal to 4. Each observation is 4 with probability 0.2, independently, so $Y \sim B(5, 0.2)$. Then $P(Y = 2) = \binom{5}{2}(0.2)^2(0.8)^3 = 10 \times 0.04 \times 0.512 = 0.2048$. (B1: binomial model $B(5, 0.2)$ stated; M1: correct binomial probability structure with $\binom{5}{2}$; A1: 0.2048.)

Question 9. Expand: $(2n+1)^2 - 1 = 4n^2 + 4n + 1 - 1 = 4n^2 + 4n = 4n(n+1)$. Now n and $n+1$ are consecutive integers, so exactly one of them is even; hence $n(n+1) = 2m$ for some integer m. Therefore $(2n+1)^2 - 1 = 4 \times 2m = 8m$, which is divisible by 8 for every integer n, completing the proof. (M1: expand the bracket correctly; A1: factorised form $4n(n+1)$; M1: argue that the product of consecutive integers is even; A1: complete conclusion $8m$, with the argument holding for all integers n, including negatives and zero.)