Name: ______________________

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IB Mathematics: Analysis and Approaches SL — Practice Exam

Instructions: Paper 1 style — NO CALCULATOR permitted. Answer all questions in the spaces provided. Show your working: method marks are awarded even when the final answer is wrong. Give exact answers (fractions, surds, multiples of pi) unless the question says otherwise.

Time Limit: 75 minutes

[6 points] 1. An arithmetic sequence has third term $u_3 = 11$ and seventh term $u_7 = 23$.

[4 points] (a) Find the first term u_1 and the common difference d.

[2 points] (b) Find S_{20}, the sum of the first 20 terms.

[5 points] 2. The functions f and g are defined by $f(x) = 2x - 3$ for $x \in \mathbb{R}$, and $g(x) = \frac{x+1}{x-2}$ for $x \neq 2$.

$$f(x) = 2x - 3, \qquad g(x) = \frac{x+1}{x-2}, \quad x \neq 2$$

[2 points] (a) Find $(g \circ f)(3)$.

[3 points] (b) Find $g^{-1}(x)$, stating its domain.

[6 points] 3. This question is about exponents and logarithms.

[3 points] (a) Solve $3^{2x-1} = 27^{x-2}$.

[3 points] (b) Show that $2\log_3 6 - \log_3 4 = 2$.

[5 points] 4. This question is about trigonometry with exact values.

[3 points] (a) Solve $2\cos x = \sqrt{3}$ for $0 \leq x \leq 2\pi$.

[2 points] (b) In triangle ABC, $AB = 6$ cm, $AC = 8$ cm and $B\hat{A}C = \frac{\pi}{3}$. Find the exact area of the triangle.

[6 points] 5. The curve $y = f(x)$ where $f(x) = x^3 - 3x + 1$ passes through the point $P(2, 3)$.

$f(x) = x^3 - 3x + 1$

[2 points] (a) Find $f'(x)$.

[2 points] (b) Find the equation of the tangent to the curve at P.

[2 points] (c) Find the equation of the normal to the curve at P.

[6 points] 6. The probability that it rains on a given morning is $\frac{1}{4}$. If it rains, the probability that Mia cycles to school is $\frac{1}{5}$; if it does not rain, the probability that she cycles is $\frac{3}{5}$. Let R be the event "it rains" and C the event "Mia cycles". A tree diagram may help.

[2 points] (a) Find $P(C)$, the probability that Mia cycles to school.

[2 points] (b) Given that Mia cycles to school, find the probability that it rained.

[2 points] (c) Determine, with a reason, whether the events R and C are independent.

[14 points] 7. The function f is defined by $f(x) = x^3 - 6x^2 + 9x$ for $x \in \mathbb{R}$. The curve $y = f(x)$ is studied throughout this question.

$f(x) = x^3 - 6x^2 + 9x$

[4 points] (a) Find $f'(x)$, and hence find the coordinates of the two stationary points of the curve.

[2 points] (b) Justify the nature of each stationary point.

[2 points] (c) Find $f''(x)$, and hence find the coordinates of the point of inflexion.

[4 points] (d) Find the exact area of the region enclosed by the curve and the x-axis.

[2 points] (e) Find the equation of the tangent to the curve at the point of inflexion.

Answer Key

1. (a) Set up two equations: $u_3 = u_1 + 2d = 11$ and $u_7 = u_1 + 6d = 23$ (M1)(A1). Subtracting eliminates u_1: $4d = 12$, so $d = 3$ (A1). Then $u_1 = 11 - 2(3) = 5$ (A1).

(b) Using $S_n = \frac{n}{2}(2u_1 + (n-1)d)$: $S_{20} = \frac{20}{2}(2(5) + 19(3))$ (M1) $= 10(10 + 57) = 670$ (A1).

2. (a) Apply the inner function first: $f(3) = 2(3) - 3 = 3$ (M1). Then $(g \circ f)(3) = g(3) = \frac{3+1}{3-2} = 4$ (A1).

(b) Write $y = \frac{x+1}{x-2}$ and make x the subject: $y(x-2) = x+1 \Rightarrow xy - x = 2y + 1 \Rightarrow x = \frac{2y+1}{y-1}$ (M1). Interchanging variables, $g^{-1}(x) = \frac{2x+1}{x-1}$ (A1), with domain $x \neq 1$ (A1) — $y = 1$ is the horizontal asymptote of g, a value g never attains.

3. (a) Express both sides with base 3: $27^{x-2} = \left(3^3\right)^{x-2} = 3^{3x-6}$ (M1). Equating exponents: $2x - 1 = 3x - 6$ (M1), so $x = 5$ (A1). Check: both sides equal 3^9.

(b) Power law: $2\log_3 6 = \log_3 6^2 = \log_3 36$ (M1). Quotient law: $\log_3 36 - \log_3 4 = \log_3 \frac{36}{4} = \log_3 9$ (M1). Since $9 = 3^2$, $\log_3 9 = 2$ (A1). This is an "answer given" part, so every law used must be shown explicitly.

4. (a) $\cos x = \frac{\sqrt{3}}{2}$ (A1). Cosine is positive in the first and fourth quadrants, so $x = \frac{\pi}{6}$ (A1) and $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ (A1). Both lie in $[0, 2\pi]$.

(b) Area $= \frac{1}{2}ab\sin C = \frac{1}{2}(6)(8)\sin\frac{\pi}{3}$ (M1) $= 24 \times \frac{\sqrt{3}}{2} = 12\sqrt{3}\,\text{cm}^2$ (A1). The answer must be left in exact surd form.

5. (a) $f'(x) = 3x^2 - 3$ (A1)(A1) — one mark for each correctly differentiated term; the constant differentiates to zero.

(b) Gradient at P: $f'(2) = 3(4) - 3 = 9$ (M1). Tangent: $y - 3 = 9(x-2)$, i.e. $y = 9x - 15$ (A1).

(c) The normal is perpendicular to the tangent, so its gradient is $-\frac{1}{9}$ (M1). Normal: $y - 3 = -\frac{1}{9}(x-2)$, i.e. $x + 9y = 29$ (A1).

6. (a) Total probability along both branches: $P(C) = P(R)\,P(C \mid R) + P(R')\,P(C \mid R') = \frac{1}{4} \times \frac{1}{5} + \frac{3}{4} \times \frac{3}{5}$ (M1) $= \frac{1}{20} + \frac{9}{20} = \frac{1}{2}$ (A1).

(b) $P(R \mid C) = \frac{P(R \cap C)}{P(C)} = \frac{1/20}{1/2}$ (M1) $= \frac{2}{20} = \frac{1}{10}$ (A1).

(c) Independence requires $P(R \cap C) = P(R)\,P(C)$. Here $P(R)\,P(C) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$, but $P(R \cap C) = \frac{1}{20}$ (M1). Since $\frac{1}{8} \neq \frac{1}{20}$, the events are not independent (A1) — rain makes Mia less likely to cycle.

7. (a) $f'(x) = 3x^2 - 12x + 9$ (A1). Stationary points where $f'(x) = 0$: $3(x^2 - 4x + 3) = 3(x-1)(x-3) = 0$ (M1), so $x = 1$ or $x = 3$. Then $f(1) = 1 - 6 + 9 = 4$ and $f(3) = 27 - 54 + 27 = 0$, giving stationary points $(1, 4)$ (A1) and $(3, 0)$ (A1).

(b) Sign of f': $f'(0) = 9 > 0$, $f'(2) = 12 - 24 + 9 = -3 < 0$, $f'(4) = 48 - 48 + 9 = 9 > 0$ (M1). The gradient changes from positive to negative at $x = 1$, so $(1, 4)$ is a local maximum; it changes from negative to positive at $x = 3$, so $(3, 0)$ is a local minimum (A1). (Equivalently, $f''(1) = -6 < 0$ and $f''(3) = 6 > 0$.)

(c) $f''(x) = 6x - 12$ (A1). $f''(x) = 0$ at $x = 2$, and f'' changes sign there (negative for $x < 2$, positive for $x > 2$), so there is a point of inflexion at $(2, f(2)) = (2, 2)$ (A1).

(d) Factorising, $f(x) = x(x-3)^2$, so the curve meets the x-axis at $x = 0$ and at $x = 3$, where it touches without crossing (M1). On $[0, 3]$ both $x \geq 0$ and $(x-3)^2 \geq 0$, so the curve lies on or above the axis and no sign correction is needed (A1). Area $= \int_0^3 \left(x^3 - 6x^2 + 9x\right)dx =$

$\left[\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2}\right]_0^3$ (M1) $= \frac{81}{4} - 54 + \frac{81}{2} = \frac{81-216+162}{4} = \frac{27}{4}$ (A1).

(e) Gradient at the point of inflexion $(2,2)$: $f'(2) = 12 - 24 + 9 = -3$ (M1). Tangent: $y - 2 = -3(x-2)$, i.e. $y = -3x + 8$ (A1).

Worked Solutions

Question 1. *(a)* Set up two equations: $u_3 = u_1 + 2d = 11$ and $u_7 = u_1 + 6d = 23$ (M1)(A1). Subtracting eliminates u_1: $4d = 12$, so $d = 3$ (A1). Then $u_1 = 11 - 2(3) = 5$ (A1).

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